Examining the Geometry Items of State Standardized Exams Using the van Hiele Model: Test Content and Student Achievement

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In this work we catalogued the content of multiple-choice geometry items on the Ohio Achievement Tests for Grades 3, 5 and 8 according to the van Hiele model of development of geometric thought. Using statewide data from 1,418 students, responses on each question were analyzed to trace students' performance at different grade levels. Statistical results indicated that the majority of the items at each grade level focused on Levels 1 and 2, and student performance declined as the question level increased. A closer examination of the participants' responses in each item suggested that visual evidence and linguistic clues significantly impacted students' judgment.

I. INTRODUCTION

Usiskin (1987) raised public concerns regarding the geometry knowledge of children in US schools and argued that except for the knowledge of shapes (something learned even before first grade), the geometry knowledge of students at the end of elementary school remains minimal. Referencing the results of the 1982 National Assessment, he pointed out that student performance remained low at all levels. He proposed that both the teaching and content of geometry taught in schools must be reconsidered so as to assure that children's learning is not hindered as a result. Nearly three decades later, results of national and international studies that measure mathematical performance of children at various grade levels continue to highlight the fragile nature of geometry knowledge of children in US schools. These results are disheartening. Geometry is a major connection between informal and formal mathematics, serving as a critical factor in student success in future mathematics classes (Duval, 1999).

In response to the disappointing results of the students' performance on norm-referenced examinations (achievement tests), some have argued that the results of these tests should not be given much weight since the test items may not be reflective of what students know and are able to do. This argument is widely used in places where students' performance on high-stakes tests is used to gauge teacher effectiveness and school ranking. In many states across the country, including Ohio, student results on standardized exams have political and financial ramifications for schools and districts. Schools are evaluated annually according to whether they have met proficiency standards and placed on "emergency" status if they fail to show progress in students' results in three consecutive years. Ultimately, schools may be shut down if their progress, as measured by standardized exams, is inadequate or insufficient. Therefore, the content of exams and validity and reliability of items used to assess knowledge are of great concern. Of particular interest in the study was the quality of geometry knowledge tested as well as student achievement on standardized exams.

II. PURPOSE OF THE STUDY

The goals of the study we report here were threefold. First, we aimed to examine the content of the geometry items used on the Ohio Achievement Tests at Grades 3, 5 and 8 according to the van Hiele model to determine what level of geometry knowledge was expected of children. The goal was to see whether the content of the tests agreed with the learning theory. Second, using data from the performance of 1,418 students from 11 schools across the state of Ohio on each of the items tested, we planned to establish a profile of geometry knowledge of the students. Lastly, by analyzing the common response choices students made on multiple-choice items, we hoped to identify factors inherent in the test items that could have contributed to students' performance. This report is part of a research project in which children's mathematical development from 85 schools across Ohio is traced over the course of three years of involvement in a statewide professional development program (Brosnan & Erchick, 2010).

III. THEORETICAL CONSIDERATION

Concern about geometry learning and teaching is not a recent development and dates back to the 1950s with the pioneering work of two Dutch teachers, Pierre van Hiele and Dina van Hiele-Geldof. The van Hieles proposed a framework that accounts for the development of geometric reasoning in order to explain how people grow in their geometry knowledge (van Hiele, 1986). They identified five different levels of understanding through which an individual passes when learning geometry, including visual, descriptive, informal deductive, formal deductive, and rigor. According to this model, these levels are not dependent on individual's physical maturation, but on the experience and instruction one accessed. While this model has been under revision by some scholars in recent years (Battista & Clement, 1992, 2007; Borrow, 2000) and criticized by some scholars for its inability to trace "in between levels of reasoning" (Burger & Shaughnessy, 1986), it is still widely used in curricula implementation in mathematics classrooms today. For these reasons we opted to use the framework as a lens for our analysis of both the tests and students' performance. A brief description of each level is presented below.

At the visualization level (Level 1), a learner identifies, names, compares, and operates on geometric figures, such as triangles, angles, and parallel lines, according to their appearance. At this level, students may see the difference between triangles and quadrilaterals by focusing on the number of sides of the polygons but are not able to identify square as one type of rectangle by their definitions.

At the descriptive level (Level 2), students can recognize components and properties of a figure using proper mathematical language, and they are able to make judgments based on definitions instead of visual appearance. However, they are not yet able to construct successive steps of reasoning upon the recognition of properties of figures, nor can they make connections among different properties and definitions.

At the informal deduction level (Level 3), students can recognize interrelationships between figures and properties, and they can justify these relationships informally. The learners can understand and use precise definitions and are capable of "if-then" language. However they are not yet conscious about the relationship between the reason and the conclusion in their argument. They may consider the reason as supportive indicators instead of decisive factors of the conclusion. They couldn't see the value of referring to a theorem or an axiom in arguing.

At the deduction level (Level 4), students can reason about geometric objects using their defined properties in a deductive pattern. They can intentionally search for the factors that could lead to a valid proof and are able to judge whether an established proof is mathematically acceptable. Students at this stage could construct the types of proofs that one would find in a typical high school geometry course.

At the highest level, rigor (Level 5), students understand the structure of logical systems and can compare axiom systems with different premises. Learners establish statements in different postulation systems and are conscious about why and when a statement holds. See an illustration of what students at each van Hiele level might use in their argument in Table 1.

Table 1. Examples of statements that students at each van Hiele level might use

van Hiele level	Example
1 Visual	"The two figures look different."
2 Descriptive	"Square is one type of rectangle."
3 Informal Deductive	"The sum of the two acute angles of a right triangle is 90 degree."
4 Formal Deductive	"Since $a \perp c$, $b \perp c$, then the congruence of corresponding angles indicates a//b."
5 Rigor	"The argument above only works in two-dimensional space."

IV. PARTICIPANTS AND METHODS

The database for the study consisted of the results of 1,418 students who had completed the Ohio Achievement Tests in May 2009. The sample consisted of 471 third-grade, 644 fifth-grade and 303 eighth-grade students' responses to the latest released math exams in their grade level. The students were from 11 low-performing elementary and middle schools. School ranking was designated by the Ohio Department of Education (ODE) according to the percentage of students who had failed to meet the "proficiency" level criteria set by the state. All participating schools were involved in a statewide professional development program that aimed to raise the mathematical knowledge of both teachers and learners in these schools.[2] Student results were compiled by each school and submitted to the research team online. Student and school identities were removed.

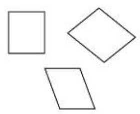
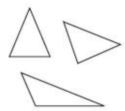
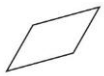
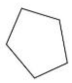
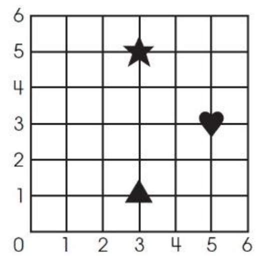
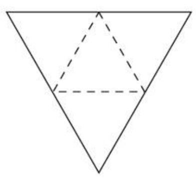
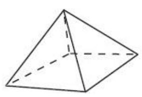
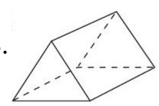
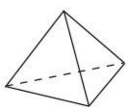
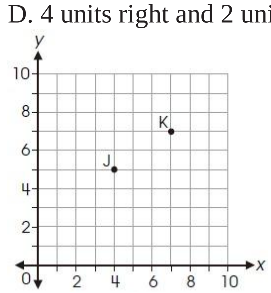
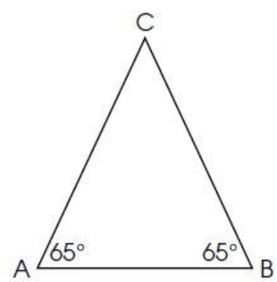
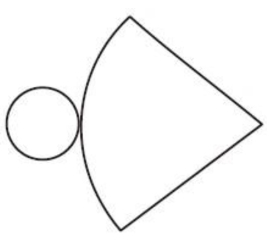
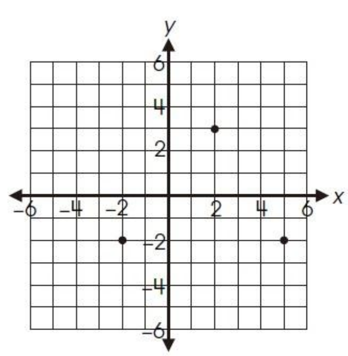
Students' responses to each multiple-choice question on each of the three grade level exams were inputted, analyzed and summarized using Excel spreadsheets. Across grade levels, the percentage of correct answers on questions in each van Hiele level was used as an indicator of students' progress.

In this work we considered only the tests' multiple-choice questions and students' responses to them in order to avoid potential inconsistencies that could exist when scoring the tests' open-ended response items, of which only one for each grade was related to geometry. Since the research team didn't score the student responses, we were not in a position to assure inter-reliability ratings of open-ended responses.

V. ANALYSIS OF THE TESTS

We identified the van Hiele level of each question by examining its content. We agreed that although higher levels of reasoning can always be adopted to solve lower-level questions, this did not change the level of difficulty of the question in itself. Hence, we used the highest van Hiele level of geometry reasoning required to solve the problem as a means to rank the level of the question on each of the tests. Additionally, we acknowledged that distinguishing one-step reasoning from identification was not always possible since this kind of reasoning is usually a deduction from definition to its sufficient and necessary condition. Nevertheless, the van Hiele level of thinking is based on experience and the instruction received (Crowley, 1987). For instance, if students are taught to identify parallel lines by using corresponding angles, a question concerning this relationship may be viewed as a Level 2 item; whereas if other ways of identifying parallel lines had been taught and the congruence of the angles was then shown to be a consequence, then this question may be classified as a Level 3 item. Therefore, without information on the actual experience of students, judging the level of question according to the van Hiele model (solely on the basis of its content) may not be adequate. However, in our analysis no subcategory between the levels was defined. Therefore, in places where disagreements occurred among the research team regarding the

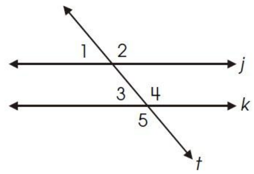
Table 2. Samples of problems in each van Hiele level

<table>
<tr><th>Grade</th><th>V.H. Level 1</th><th>V.H. Level 2</th><th>V.H. Level 3</th></tr>
<tr><td>3</td><td>Two groups of shapes are shown below. Which shape belongs in Group 2?
Group 1: 
Group 2: 
A  B.  C. </td><td>What is the location of the star on the grid?
A. (3, 1)
B. (3, 5)
C. (5, 3)
</td><td></td></tr>
<tr><td>5</td><td>A net of a three-dimensional shape is shown. Which three-dimensional shape can be made from the net?

A.  B. 
C.  D. </td><td>Point J and point K are shown on the grid. What is the direction from point J to point K along the grid lines?
A. 3 units right and 2 units up
B. 3 units right and 3 units up
C. 4 units right and 3 units up
D. 4 units right and 2 units up
</td><td>Triangle ABC is shown. What is the measure of angle C?
A. 50° B. 65°
C. 90° D. 180°
</td></tr>
<tr><td>8</td><td>Ray found the paper cut-out shown. Which 3-dimensional object is formed when the cut-out is assembled?
A. cone
B. cylinder
C. prism
D. sphere
</td><td>Three vertices of a trapezoid are located at points (2, 3), (–2, –2) and (5,–2). Which point could represent the fourth vertex of the figure?
A. (5, 0) B. (5, 4)
C. (–1, 3) D. (–1, 5)
</td><td>Circle A has a radius that is twice the length of the radius of Circle B. Which is an accurate statement about the relationship of the areas of Circles A and B?
A. The area of Circle A is four times the area of Circle B.
B. The area of Circle A is twice the area of Circle B.
C. The area of Circle A is one-half the area of Circle B.
D. The area of Circle A is one-fourth the area of Circle B.</td></tr>
</table>

level of a question, we identified the question to be in the "closer" level by researchers' judgment.[3] For example, consider the following example from the eighth-grade test:

(Eighth grade) In the figure, lines j and k are parallel. Which angle is congruent to $\angle 1$?

A. 2 B. 3 C. 4 D. 5



While it could be argued that the above question measures Level 3 thinking since the congruence of $\angle 1$ and $\angle 3$ is a conclusion following the statement of parallel lines, i.e. "If j and k are parallel, then $\angle 1$ and $\angle 3$ are equal", asserting then that reasoning from lines to angles is involved. Nevertheless, it is also legitimate to argue that congruence of $\angle 1$ and $\angle 3$ can be viewed as an identifier of parallel lines; hence the question can be ranked as an "identification" task instead of reasoning that leads to the solution. In this case, we classified the question as Level 2 since we believed one-step reasoning to be "closer" to description than to relation. In Table 2, we exemplified problems that were classified in each van Hiele level.

Table 3 offers a blueprint of the test items at each grade. As illustrated, all but one question on the third-grade test are in Level 1. These items measure knowledge of triangles, quadrilaterals and the number of their sides and angles. The only Level 2 question on the third-grade test refers to a basic identifier, i.e. convention to describe locations in the coordinate grid. No other properties of the grid, such as the parallel or perpendicular lines, are involved in the question.

In 1992, Clements and Battista concluded that, in geometry, students were extremely unsuccessful with formal proof (upper Level 3 and Level 4). We noticed that (as shown in Table 3), tests were mostly measuring low levels of knowledge. More than 85% of the test items are designed to measure Level 1 thinking in Grade 3, and a large percentage of questions in fifth- and eighth-grade tests were also ranked at Levels 1 and 2 (75% and 67%, respectively).

Table 3. Questions grouped by and van Hiele level

Grade	# of Lv. 1 questions	# of Lv. 2 questions	# of Lv. 3 questions	Total # of questions
3	6 (86%)	1 (14%)	0	7
5	2 (25%)	4 (50%)	2 (25%)	8
8	1 (17%)	3 (50%)	2 (33%)	6

The van Hiele levels of questions used on the fifth-grade test were generally higher than those on the third-grade test. The two Level 1 questions asked students to examine three-dimensional shapes. The figures were more sophisticated to visualize than two-dimensional figures when drawn on paper. Therefore, the problems required higher visual ability to solve compared to the third-grade items although were both classified as Level 1 questions. The four Level 2 questions asked students to find the relative location of lines, symmetry of a circle by its diameter, and geometric properties in the coordinate. These concepts were more advanced than the concepts of sides and angles since they are described by more advanced geometric concepts. The two Level 3 questions measured students' understanding of the interior angle sum formula of a triangle. Students were asked to find the degree measure of an angle given the measures of the other two angles.

On the eighth-grade test, only one Level 1 question was noted and it required three-dimensional thinking. The three Level 2 questions measured knowledge of parallel lines' angle-related properties, identification of figures and their transformation in the coordinate plane. The two Level 3 questions were also more sophisticated than Level 3 questions in the fifth-grade test. One question tested students' perception of the similarity of triangles and required calculating the index of proportionality of figures. The second question required the use of algebra in calculating the area of a triangle.

As the blueprint illustrates, the van Hiele levels of the questions asked on the three tests increased regarding grade level. Therefore, the design of the geometry questions on the achievement tests we studied was consistent with the developmental sequence of geometry reasoning according to the van Hiele model. Nonetheless, the significant portion of each test at each grade level focused on low levels of geometry reasoning. We will address the significance of this issue in the discussion section of this article.

VI. ANALYSIS OF STUDENTS' PERFORMANCE

Tables 4-7 summarize the percentage of students who chose the correct response on each geometry question on each of the achievement exams. A decrease in the percentage of correct responses to higher-level questions compared to correct lower-level responses within every grade level is detectable. In the van Hiele model, as with most developmental theories, a student must proceed through the levels in order, and to function successfully at a particular level, a learner must have acquired the strategies of the preceding levels (Crowley, 1987). Therefore, students who reached higher levels would be fewer than those who only achieved a lower level. Thus, the decrease in performance is consistent with the sequence of development in van Hiele's theory.

Table 4. Third-grade students' percentage of correct responses to each geometry question

V.H. level of question	L1	L1	L1	L1	L1	L2	L1
% of correct responses	87%	26%	87%	76%	40%	51%	50%

Table 5. Fifth-grade students' percentage of correct responses to each geometry question

V.H. level of question	L2	L1	L3	L2	L3	L2	L1	L2
% of correct responses	46%	44%	28%	28%	32%	36%	47%	32%

Table 6. Eighth-grade students' percentage of correct responses to each geometry question

V.H. level of question	L2	L3	L1	L3	L2	L2
% of correct responses	51%	50%	67%	24%	48%	48%

Table 7 illustrates the average percentage of correct responses for questions from each level at each grade accordingly and in all the three grades combined together.

Table 7. Average percentage of correct responses by grade and van Hiele level

Grade	Level 1	Level 2	Level 3
3	61%	51%	NA
5	46%	36%	30%
8	67%	49%	37%
All	55%	44%	32%

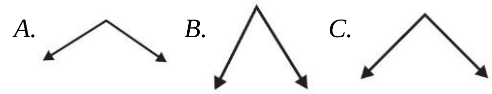
The third graders' average percentage score for the 6 items which were ranked at Level 1 of geometric reasoning was 61%. This average score decreased to 51% for Level 2 questions. Similarly, the fifth graders' average percentage scores on Levels 1, 2 and 3 items were 46%, 36% and 30%, respectively. Lastly, eighth graders' average scores declined from 55% to 44%, and 32% as the level of question increased from 1 to 3. For the entire sample, the average percentage score on Level 1 items was 55%. This number declined to 44% for all items ranked as Level 2. Finally, the average percentage score for Level 3 items was 32%. The results indicate low student performance in all three levels. More importantly, as a group, the performance also declined as the level of geometric thinking increased on the tests, which was consistent with the van Hiele's theory.

In order to better understand the results, a close examination of items and factors that could have influenced children's choices of wrong answers was conducted. We particularly focused on analyzing students' representation of the problem, which is the essential starting stage of problem solving activities. After having done so, we propose three conjectures regarding students' performance based on the language and form of the questions used. The conjectures were consistent with findings from existing studies (Lesh & Doerr, 2003; Montague, 2003; Huang & Normandia, 2008).

Conjecture 1: Inferences can be drawn based on past experiences and concept images developed during instruction.

As an example, let us consider an item from the third-grade test. The question asked students to identify a picture that showed a right angle with the following images provided:

(Third grade) Which picture shows a right angle?

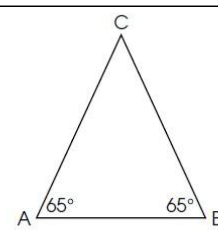


Among the 471 third graders, 171 chose "A," 174 chose "B", 123 chose "C" as their answer (3 students didn't offer an answer). Small differences among the total students who made each choice indicate they were confused since none of the choices looked like what they normally saw as a right angle (in the visual level). We argue that since teachers usually draw a right angle in their instruction as the intersection of a vertical ray and a horizontal one, the students failed to see the picture turned by 45° as the same figure.

Another similar example was noted on the fifth-grade test. The problem asked students to identify the measure of the missing angle in the image when the measures of two other angles were given, as shown below.

(Fifth grade) Triangle ABC is shown. What is the measure of angle C?

A. 50° *B. 65°*
C. 90° *D. 180°*



The most common choice was "B", selected by 202 of 644 (31%) fifth-grade students whose responses were examined. It is plausible that the students assumed the figure represented an equilateral triangle (in the visual level) and every angle of it should be equal accordingly. This image is consistent, again, with what is frequently used in class in teachers' demonstrations of concepts. The imagery could have evoked strong concept images, directing students to incorrect conclusions.

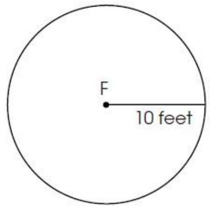
Conjecture 2: Inferences might be drawn due to linguistic clues, i.e. appearance of numbers more than once in the context of the problem.

In considering the fifth graders' common response to the test item that asked them to identify the length of the diameter of a circle given the measure of the radius, 34% of the 644 students selected "10" as their response option. Note that in this question the numeral “10” appeared in both the figure and also in the option “A.”

(Fifth grade) Point F is the center of the circle shown. What is the diameter of this circle?

A. 10 feet *B. 20 feet*

C. 30 feet *D. 100 feet*



When considering the eighth graders’ responses to the question below which asked them to find the ratio of the areas of two circles with the radius of one circle twice the radius of the other, the common response option “B” selected by 42% of the sample can be explained in a similar manner. In this question “twice” appeared both in the condition and in option “B.”

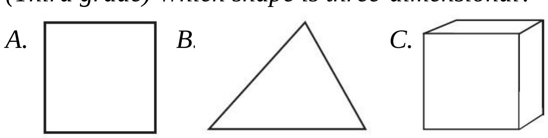
(Eighth grade) Circle A has a radius that is twice the length of the radius of Circle B. Which is an accurate statement about the relationship of the areas of Circles A and B?

A. The area of Circle A is four times the area of Circle B.
B. The area of Circle A is twice the area of Circle B.
C. The area of Circle A is one-half the area of Circle B.
D. The area of Circle A is one-fourth the area of Circle B.

Conjecture 3: Inferences might be drawn due to linguistic clues or meanings students attached to words from personal experiences.

In elaborating on the influence of language on students’ choices, let us consider two examples from third- and fifth-grade achievement tests, as shown below.

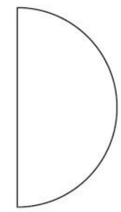
(Third grade) Which shape is three-dimensional?



Note that in response to the first example (third-grade test: “Which shape is three-dimensional?”), 53% of the third graders selected “B” (the triangle) as the response option. In retrospective, students could have associated the “three” in “three-dimensional” with “three” as in “three-sided figure.” In this context, students could identify the triangle (option “B”) as the object with “three” sides. This knowledge was previously tested with the use of another question on the same test that 87% of the sample answered correctly. The language of the text could have provided the wrong hint for selection of the response. The recall from that context could have certainly influenced the children’s choice in this problem space.

In reading and interpreting mathematical problems, students draw from multiple resources including their own experiences from real life and how terms are used in their daily lives. An example of such an influence is evidenced in the fifth graders’ popular response to the problem below.

(Fifth grade) Malcolm needed to measure the distance across a circular tablecloth. He folded the tablecloth in half as shown. Malcolm measured the length of a folded side. Which part of the circular tablecloth did Malcolm measure?



A. center *B. circumference*
C. diameter *D. radius*

The most commonly selected response option to this question was “A” (center). This option was chosen by 222 (34%) of the fifth graders in our sample. The student might have interpreted the folding of the circle in half as finding the center of the circle. That is, the folded side is in the middle of the circle, dividing the tablecloth into two equal parts; hence, it is the center of the tablecloth. We conjecture that such contextual interpretation may have contributed to the students’ choice, assuming middle to be the center.

Our data are certainly limited both in quality and quantity to permit conclusive inferences regarding children’s thinking or influences that could have impacted their choices. We agree that more detailed, descriptive data might provide some direction in better understanding the contributing factors to the children’s choices.

VII. DISCUSSION

Analysis of students’ performance at three different grade levels on geometry items used on Ohio Achievement Tests clearly speak to problems associated with school geometry learning. Students in all three grade levels had difficulty with visual identification of geometric concepts. Indeed, as specific features were added to a figure, students’ performance declined by a much larger margin. Collectively, the students’ perception of geometric concepts was underdeveloped at both the visual and descriptive levels, and the weak foundation would definitely impede the development of reasoning skills at higher levels. Data further indicated that students had difficulty recalling mathematical terms and definitions.

There is no doubt that student performance on standardized exams is influenced by a variety of instructional and non-instructional factors. From the point of view of instruction, there is always the potential for existence of a mismatch between what is tested and what is taught in the classroom. It is certainly plausible that performance on items that test children’s knowledge of basic facts may not be as high when instructional focus is on inquiry and conceptual development. Hence, while we are cautious about making general instructional recommendations based on the findings from this study, we posit that considering that we examined geometry knowledge of the sample using a well-established theory of learning geometry, the results should be considered independent of specific instructional contexts.

Analysis of the test items make explicit the need for a careful consideration of what is included on tests, both in language and content, in order to adequately measure student development. This is particularly important since a "teaching to the test" framework can limit children's mathematical experiences in classrooms, focusing teachers' attention to only lower cognitive level tasks. Analysis of the test items and student achievement also highlight the need to devote greater attention to how geometry is taught in schools.

ENDNOTES & REFERENCES:

1. An earlier version of this study was presented at the 2010 PME-NA annual conference, Columbus, OH. Email Yating Liu at liu.891@buckeyemail.osu.edu, to whom correspondence concerning this article should be addressed.
2. For a detailed description of the project see Brosnan & Erchick (2010).
3. We concur that, regardless of how carefully we may define the levels, debates on their borderlines always exist. As a result, findings of the study should be considered with respect to these flexibilities.

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